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MODAL STIFFNESSES OF A PRETENSIONED CABLE NET

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Abstract—The paper is concerned with the structural mechanics of pretensioned saddle-shaped cable nets of the type shown in Fig. 1. Such nets are normally regarded as being non-linear systems; but it is shown that their behaviour may be described satisfactorily in terms of two distinct, and practically independent, sets of *extensional* and *inextensional* modes. Each set of modes may be studied by means of suitable *linear* analysis, and the eigenmodes may be found without difficulty. The stiffness of the extensional modes derives from the elasticity of the members, and is practically independent of the level of prestress in the net. In contrast, the stiffness of the inextensional modes is independent of the elasticity of the members; it derives from geometry-change effects and is directly proportional to the level of prestress in the net. Usually, the most compliant inextensional mode of a net made from steel cables will be much less stiff than the extensional modes; but the circumstances of this are determined by the value of a single dimensionless group. An experiment performed in the laboratory on a small-scale model net confirms the theoretical results. The paper concludes with a short discussion on the applicability of the results of this paper to pretensioned cable nets in general.

1. INTRODUCTION

The aim of this paper is to elucidate the structural mechanics of saddle-shaped cable nets. Figure 1 is a schematic view of such a cable net: it shows the two sets of elastic cables, slung between rigid abutments, which are initially prestressed against each other when the net is free from external loads.

How does the assembly respond when arbitrary external loads are imposed upon it? How does the assembly deform, and how do the tensions within the system change in response to external loads?

The conventional answer to these questions is conditioned by the frequent assertion in the extensive literature (e.g. [1–3]) that cable nets are *non-linear* systems, since in general the equilibrium equations must be set up with respect to the deformed configuration. It then follows that there are no simple answers to these questions; and if computations are to be done they must be performed numerically by means of powerful non-linear computer routines.

The main object of this paper is to show that, on the contrary, the behaviour of a cable net under load can be described in a relatively straightforward and simple way, at least as a first approximation. The key to the situation is that cable nets of the type shown in Fig. 1 have two distinct modes of action; and each of these may be understood in terms of the behaviour of a separately *linear* system. Once this point has been grasped, it is a relatively simple matter not only to calculate the response of the cable net to various types of imposed loading—at least as a first approximation—but also to determine the eigenmodes and simple formulae for the

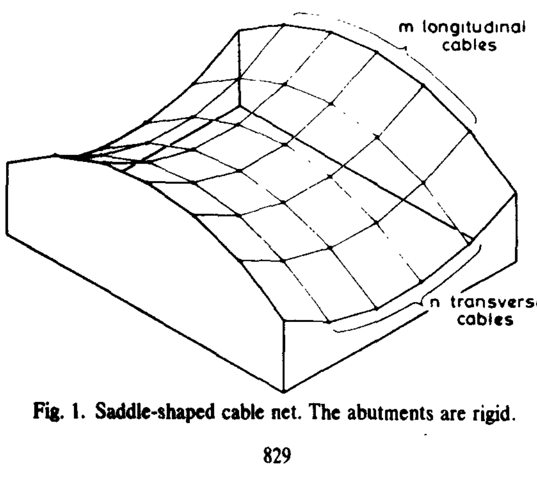

Fig. 1. Saddle-shaped cable net. The abutments are rigid.

corresponding stiffness of the net in its two distinct modes of action. But the aim of the paper is not merely to catalogue results which may be obtained in this way. In many practical nets the stiffnesses of the system in the two basic modes of action will differ by about an order of magnitude; and the paper explores the dimensionless group which determines the magnitude of this factor.

For the sake of definiteness we shall use a specific example, as shown in Fig. 1. And in order to simplify the necessary calculations we shall further regard the net as *shallow*, using this term in the way in which it is widely used in the theory of shell structures [4].

There is, of course, a danger that by restricting attention to a specific example and then making some simplifying assumptions, we may rob ourselves of the possibility of making observations which are generally applicable. But it seems clear that the main features of behaviour of cables nets survive this process (see Section 13) in much the same way that the main features of shell theory may be studied, at least qualitatively, by means of the shallow-shell approach.

The analysis is entirely statical throughout. Nevertheless, the modal stiffness formulae which are developed may readily be adapted for dynamical purposes once the mass of the system has been specified. The analysis disregards any structural effects which may be due to the application of cladding to the net.

The layout of the paper is as follows. After some preliminaries in Section 2, Section 3 uses well-known results of linear algebraic analysis to determine the numbers of inextensional and extensional modes, respectively, for a given net. Section 4 introduces the useful idea of the "plane comparison net", which will facilitate some later calculations, and Section 5 describes the kinematical features of the inextensional modes of a curved net. Sections 6–8 are concerned with the stiffness of extensional modes of a curved net and the stiffness of inextensional modes of plane and curved nets, respectively: in each case the eigenmodes and their corresponding stiffnesses are determined. Section 9 shows that the "softest" inextensional mode is normally much more compliant than the various extensional modes, and investigates the circumstances in which the two kinds of mode may have stiffness of the same order. In Sections 10 and 11 two loose ends from the preceding work are tidied up. Section 12 describes a simple experiment which confirms the analytical work, and the paper concludes with some general remarks in Section 13.

2. DESCRIPTION OF A CABLE NET

Before we can begin the analysis we must describe the system and introduce some nomenclature. The net consists of m (≥ 2) cables running in one direction and n (≥ 2) cables running in the other: see Fig. 1. All of the cables lie in vertical planes. They are securely attached to rigid abutments at their ends, and they are firmly connected together at the nodes where they intersect. It will be convenient to refer arbitrarily to the family of m "hanging" cables which are concave upwards as the *longitudinal* cables, and to the other family of n "bracing" cables as the *transverse* cables.

For the sake of simplicity in the subsequent analysis we shall regard the distances along all cables between successive nodes as being roughly equal; and we shall also assume that the small angles turned through by the cables at the nôdes are approximately equal for all cables. There is no difficulty in the adaptation of the detailed formulae of the present paper to nets whose two sets of cables have different characteristics.

In practice, of course, the cables of a net are continuous. For the purpose of analysis, however, it is more convenient to regard the cables as strings of straight *bars* which are freely pivoted at the nodes and the abutments, by ideal frictionless *joints*. The imposition of frictionless joints is a satisfactory idealisation of an extremely slender member such as a cable.

For some purposes it will be satisfactory to regard the bars as being *rigid*, or *inextensional*. But for other purposes it will be necessary to take into account the *elasticity* of the bars: we shall do this by relating the elongation e of a bar (with respect to its initial, prestressed state) to the change in tension ΔT by means of the elastic *stiffness* AE/l:

$$\Delta T = (AE/l)e. \tag{1}$$

Here A, l and E are, respectively, the cross-section area and length of the bar, and the Young's modulus of the material of which the bar is made. It will be satisfactory for present purposes to adopt the same value of (AE/l) for every bar of the net.

In general we shall not refer hereafter to the constituent *cables* of the net, but to the *strings* of the bars into which we idealise it. We shall also refer to the joints as "nodes".

Sometimes we shall refer to an external force or load W which is applied to a joint or mode in the *normal* direction. By this we shall mean normal to the plane which passes through the node and is closest, in the r.m.s. sense, to the four neighbouring nodes. But for most purposes, as indeed in the "shallow shell" theory, there is no crucial difference between the "normal" and "vertical" directions.

We shall describe the distortion or deformation of the net mainly by means of the set of "normal" components w of displacements of the joints. The positive sense of both W and w is usually *downwards*.

3. THE CABLE NET AS A PRETENSIONED MECHANISM

In general we must expect that the application of a set of loads W at the nodes will be resisted to some extent by changes of tension in the bars; that consequently there will be elastic changes of length of these bars; and that these extensions will contribute towards the displacements of the joints.

Such an expectation is misleading in the present case, although it cannot be faulted in relation to many conventional structures composed of rods and joints. The assembly under consideration is not a *structure* in the ordinary sense, but a pre-tensioned *mechanism* with a number of degrees of freedom of order $n\,m$. More specifically, the assembly can undergo small distortions in which the lengths of the members do not change; and these "infinitesimal" distortions are known as *inextensional* modes.

Linear algebra is the proper tool for studying the existence of such modes in an arbitrary assembly of rigid (inextensional) rods and frictionless joints: see [1], Section 4.1, and [5]. Thus, a combination of the equations of equilibrium of the assembly with the equations of kinematics of small distortion by means of the principle of virtual work leads to the general result that in an assembly of b rods connected to each other at j joints and to an arbitrary extra number of points on a rigid foundation,

$$3j - b = f - s. \tag{2}$$

On the r.h.s. of this equation f represents the number of *degrees of freedom* ($\geqslant 0$) of the assembly as a mechanism, while s is the number of *independent states of self-stress* ($\geqslant 0$) in which the members may be in a state of tension while the joints carry zero external load.

In relation to the assembly of Fig. 1,

$$j = mn \text{ and } b = n(m+1) + m(n+1)$$

by inspection; and so

$$f - s = mn - m - n. \tag{3}$$

In the present case it is easy to show that there is precisely *one* state of self-stress: for if the tension in any one bar is given, the tension in every other bar may be found by using the equations of equilibrium of the (unloaded) joints in turn. (The fact that there are apparently more equations of equilibrium than are needed for this purpose will be discussed briefly in Section 13.) Thus

$$s = 1, \tag{4}$$

and hence, from (3)

$$f = (m-1)(n-1). \tag{5}$$

This gives the number of degrees of freedom of the infinitesimal modes for the assembly shown in Fig. 1.

Now in general, bar-and-joint structures assembled from bars of *arbitrary* length usually fall into one of three classes, which may be described as follows:

(a) *redundant structures* having no kinematic freedom ($f = 0$) but several states of self-stress ($s > 0$),

(b) *Mechanisms* having several degrees of freedom ($f > 0$) but no states of self-stress ($s = 0$),

(c) *statically determinate, just-stiff structures* having neither kinematic freedom nor states of self-stress ($f = 0$ and $s = 0$). Equations (4) and (5) indicate that the present assembly falls into none of these usual categories. It is an example in which both $s > 0$ and $f > 0$: there are simultaneously non-zero states of self-stress *and* non-zero degrees of freedom. Apart from trivial cases where distinct *parts* of an assembly lie in classes (a) and (b), respectively, this fourth class occurs only when the lengths of the bars satisfy certain restrictions [5]. Thus, in the present example it would not be possible to connect the assembly of rigid bars together if any one bar were any shorter than it actually is: and it seems clear intuitively that this feature leads to the possibility of sustaining a state of self-stress in the "tight" assembly. In contrast, on the other hand, if any one bar were made *longer* than it actually is, the assembly would become "loose": the value of s would then become zero and the value of f would be 1 greater than that given by (5), and the assembly would revert to class (b).

The preceding remarks are based firmly on the supposition that every rod is strictly inextensional. The rods of the actual assembly are all elastically deformable, and we may therefore argue that onto any individual joint may be imposed independently a small, arbitrary normal component of displacement. In this sense the assembly has a total of mn degrees of freedom. We thus find that since $(m-1)(n-1)$ of them are purely inextensional, as demonstrated above, the remainder must involve some extension of the bars. Hence we conclude that the assembly has two distinct types of mode of normal deformation:

$$\left.\begin{array}{c}(m-1)(n-1) \text{ inextensional modes} \\ \text{and} \\ m+n-1 \text{ extensional modes.}\end{array}\right\} \tag{6}$$

Our main task in the remainder of the paper will be to study separately the mechanics of the two different types of mode of deformation. We shall investigate what types of loading pattern are associated with each of the two families of distortion, and the corresponding quantitative relationships between load and displacement.

4. MECHANICS OF A PLANE "COMPARISON NET"

It is useful to pursue our study of the cable net of Fig. 1 by considering next the plane "comparison net" which is shown in Fig. 2. Like the curved net, it consists of m longitudinal cables intersecting n transverse cables and connected to rigid abutments; but in this case all of the cables are straight and lie in a single plane. In particular, the numbers of bars and joints are exactly the same as before.

Now in this plane net every straight cable may sustain independently an arbitrary tension, without the equilibrium condition for the joints of the unloaded net requiring tension in any other member. Thus we find that for the tight, plane net

$$s = m + n. \tag{7}$$

Therefore, in view of the general relationship (2) the number of inextensional modes (all infinitesimal) in this case is given by:

$$f = mn. \tag{8}$$

Thus *all* of the mn independent out-of-plane modes are inextensional for the plane net: there are no out-of-plane *extensional* modes for this type of net.

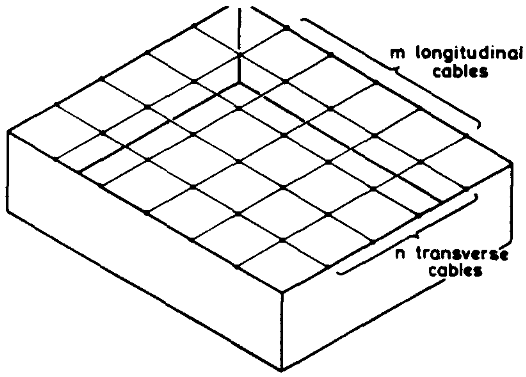

Fig. 2. Plane "comparison net", having the same cable layout as the saddle-shaped net. The abutments are rigid.

The plane comparison net enables us to clarify several points in relation to the original curved net, as follows.

First, the inextensional modes are easy to visualise, since each node can undergo a small displacement normal to the net without imparting significant strain to the adjoining members. Of course, there must be strictly some change of length in any operation of this sort, since the straight line is the shortest distance between any two fixed points. The important point here is that the strains incurred are of order $(w/l)^2$, where l is the length of a bar; and since this is negligible provided the value of w is sufficiently small, it is appropriate to call the modes "infinitesimal". In contrast, the strain incurred in the members of an ordinary redundant or statically determinate structure (classes (a) and (c), above) are of order w/l when displacement w is imposed on a given joint. Similar remarks apply to the "inextensional" modes of the curved cable-net: for small displacements these incur negligible strain.

Second, it is obvious in the plane net that if the condition $w = 0$ is imposed on every node, the assembly still has $2mn$ degrees of freedom in relation to *in-plane* displacement of the joints. These modes plainly require changes in length of the bars, and are therefore clearly not inextensional: when loads acting *in the plane of the net* are applied to the nodes, the net obviously responds in terms of in-plane modes and not in terms of out-of-plane modes. In contrast, when loads are applied tangentially to a curved net, in general there will occur some displacement of joints in the normal direction. The big difference here between plane and curved nets is that in general the nodes of a curved net must be restrained kinematically from normal displacement if purely tangential displacements are to occur; and such kinematic restraints would in general induce normal *reactions* to the net in response to arbitrary tangential loading. In this paper we shall not be concerned with purely tangential displacements of curved cable nets. These artificial, constrained modes correspond broadly to Lamb's "first class" of "wholly tangential" modes of displacement of shells when normal displacements are forbidden[6].

5. INEXTENSIONAL MODES OF CURVED NETS

The striking difference between the number of inextensional modes in the curved net and the plane net ($(m-1)(n-1)$ and mn, respectively) which we have demonstrated by application of the general relation (2), may be demonstrated by kinematic arguments in the present examples by means of the simpler two-dimensional inextensional modes of straight and curved strings of bars, respectively. Figure 3(a) shows a straight string of bars connected by n joints between rigid abutments. A single joint of this assembly may be given a small transverse displacement, which induces only second-order extensions of the bars, as we have seen. There are clearly n independent modes of this kind, one for each of the interior nodes.

Consider now the uniform plane "curved" string of rigid bars shown in Fig. 3(b). There are $n+1$ equal bars, each of length l, and in the original curved configuration the small angle α between adjacent bars is equal at all nodes. If we attempt to give a small (in-plane) normal displacement w to a single node (while keeping $w = 0$ at all other nodes) we shall be unable to

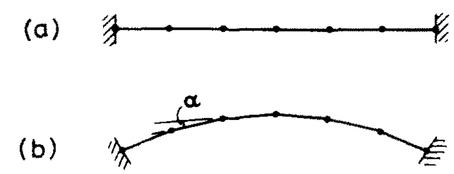

Fig. 3. (a) straight and (b) curved strings of rigid bars between fixed abutments and connected at n interior nodes. Displacements are confined to the plane of the diagram.

do so if the bars remain rigid. But if we now disconnect the pin at the joint which is being displaced, the move becomes possible, and by elementary kinematics we find that a small gap of αw opens up at this joint. And indeed, a gap of essentially the same amount would have opened up at any other joint if the single disconnection had been located elsewhere.

It follows directly that if one joint in a plane unbroken chain of rigid bars as in Fig. 3(b) is given a small outward displacement w, this can certainly take place if any other joint is given an inward displacement of the same magnitude. And moreover, this result may be generalised into the condition

$$\sum_1^n w_i = 0 \tag{9}$$

for the inextensional displacement of the uniform plane chain of Fig. 3(b) subjected to small normal displacements $w_i (i = 1...n)$ at the n joints. The coefficients of w_i implicit in (9) are equal (and hence unity) if and only if the angles α between the bars are all equal.

It follows immediately from this that the curved chain of rods has $n-1$ degrees of freedom under small displacements (since condition (9) must be satisfied), in contrast to the n degrees of freedom of the straight string. Furthermore, we can see by an extension of this argument that the doubly curved cable net of Fig. 1 has $(m-1)(n-1)$ degrees of freedom, since small displacements may be applied at $(n-1)$ joints of $(m-1)$ of the longitudinal strings. One longitudinal string must be left free to satisfy the condition (9) for the transverse strings, and the single remaining node must adopt a value of w which sets the global sum of displacements to zero. We have thus verified (6) for the curved net by purely kinematic reasoning.

Consider next an inextensional mode of the curved net in which a certain joint is given a positive displacement w. This joint, in particular, lies on two strings. Suppose that the condition (9) for each of these strings separately is met by giving one other joint a compensating negative displacement. Then the condition (9) for each of the other two strings which pass through these two joints may be satisfied by having a positive displacement at the joint where these two strings intersect. This pattern of displacement, involving normal displacements of equal magnitude but alternating sign at the four corners of a rectangle, and shown schematically in Fig. 4(a), may be regarded as an elementary inextensional mechanism of the system. Hence, if this quadrangle of displaced joints is shrunk to an elementary square (Fig. 4b), we can see immediately that there are exactly $(n-1)(m-1)$ independent inextensional mechanisms of the system. Again this agrees with (6).

6. EXTENSIONAL MODES OF THE CABLE NET

Our next task is to investigate the *extensional* modes of the cable net. It is most satisfactory to do this by considering the response of the net to certain definite patterns of loading. As we have seen already, there are $m+n+1$ independent extensional modes. How may these be characterised?

It is clear that if a pattern of loading is to produce a purely extensional mode, it must be incapable of exciting any of the $(m-1)(n-1)$ inextensional modes: that is, the pattern of loads must be orthogonal to every inextensional mode. The simplest loading set which satisfies this requirement is that in which only one string, say a longitudinal one, is loaded by equal normal forces applied to every joint of it. This remark may be verified by inspection. Let us call this type of equal loading of all joints on a single string *uniform* loading of the string. Furthermore,

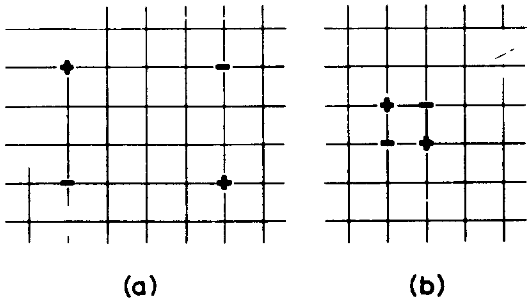

Fig. 4. Schematic representation of inextensional modes for a net like Fig. 1. The nodes labelled + move upwards and those labelled − move downwards. Other nodes do not move (normal to the net). The ends of the cables are fixed to rigid abutments. (a) A general case. (b) A compact case.

it is clear that no inextensional mode will be excited if each longitudinal string is loaded uniformly, but independently. Such a loading scheme has m independent parameters, i.e. one for each string. The same argument applies equally, of course, to uniform loading of the n transverse strings. Thus there are, apparently, altogether $m+n$ disposable parameters in the description of loading patterns which are orthogonal to the inextensional modes. But this is contrary to our previous result that there are $m+n-1$ independent extensional nodes. The paradox is resolved by the remark that the unique loading pattern in which equal loads are applied at every one of the nm nodes may be regarded *either* as a set of uniformly loaded longitudinal strings *or* a set of equally loaded transverse strings. It is one loading pattern, not two; and it follows immediately that the number of independent loading parameters is actually $m+n-1$, in agreement with our previous result (6).

Let us now investigate the way in which the net responds to uniform loading of a single longitudinal string by equal loads W applied at every node.

In Fig. 5 this string has been shown detached from the net. Each node of the string is shown to carry a fraction $(1-\beta)$ of W, while the corresponding node of the other part of the net carries the remainder. The fraction β is unknown as yet, and we shall determine its value by a kinematic matching of the displacement of the string and the remainder of the net. We have already argued that this type of net has a single redundancy ($s=1$); and the parameter β represents this redundancy for the loading under consideration.

The tension in each bar of the net may be calculated by means of the equations of equilibrium. Thus, when changes of geometry on account of loading are ignored, the tension in each bar of the loaded string is equal to

$$(1-\beta)W/\alpha, \tag{10}$$

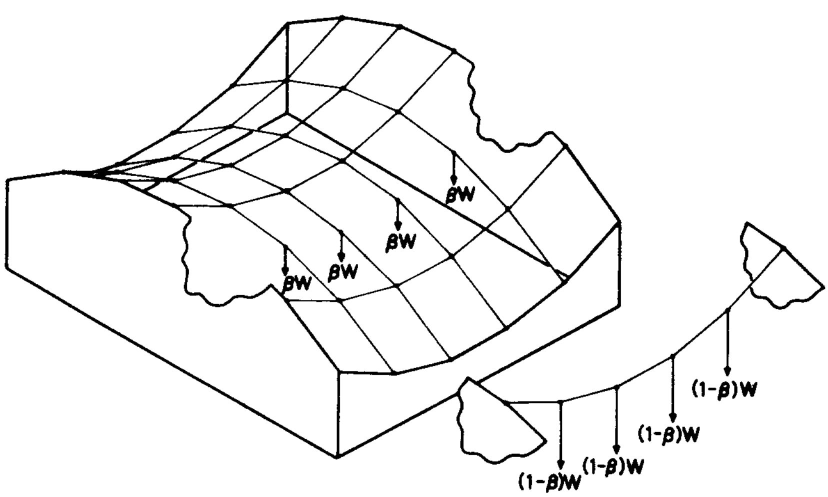

Fig. 5. Isolation of a single longitudinal cable for the determination of the stiffness of an extensional mode.

and the tension in each bar of the remainder of the net is equal to

$$-\beta W/\alpha. \tag{11}$$

The latter follows from the equations of equilibrium of the loaded joints of the remainder, together with the observation that no other joint carries any external load.

These tensions are all in addition to the original pretension in the assembly, which we shall assume to be greater in magnitude than the compressive force which any bar is called upon to sustain on account of the applied loads W. The change in extension of every bar in response to the applied load is given by (1).

The next task is to obtain an expression for the kinematic compatibility of the loaded cable and the remainder of the net. In this we shall employ the principle of virtual work. At first it may seem obvious that we must match the displacement of every node of the loaded string separately to that of its counterpart in the remainder of the net. But such a procedure would generate n expressions for the determination of the single parameter β. And indeed, if we were to attempt to find the displacement of a single node of *either* the loaded string *or* of the remainder of the net, we would immediately run up against the problem that the application of a single unit load to a joint (for the purposes of calculation of a displacement by means of virtual work) would excite one or more of the inextensional modes of the system. The only way of avoiding entirely the excitation of inextensional modes is to apply a unit load to every joint of the loaded string: hence, after all, we can obtain only one condition of compatibility.

When a unit load is applied to every joint of the isolated longitudinal string, the tension in every member of the string is $1/\alpha$. Hence, on application of the principle of virtual work we find

$$\sum w = (n+1)(1-\beta)\, Wl/AE\alpha^2: \tag{12}$$

the summation is for the n joints of the loaded string.

A similar calculation for the remainder of the net gives

$$\sum w = (2nm + m - 1)\beta Wl/AE\alpha^2: \tag{13}$$

the first term on the r.h.s. is the number of bars in the remainder of the net. Hence, on putting the two expressions together, we find

$$\beta = (n+1)/(2mn + m + n). \tag{14}$$

Having determined the value of β in a given case we may then compute Σw for the loaded string, and also, by a further application of virtual work, Σw for any other string of the net.

When $m \gg 1$ and $n \gg 1$ (14) reduces to the approximate relation

$$\beta \simeq 1/2m. \tag{15}$$

This expresses the fact that the compliance of the remainder of the net is proportional to the number of other cables parallel to the single loaded cable; and the factor 2 corresponds to the fact that the rods in both sets of strings contribute equally to the compliance of the remainder. Formula (14) may be used in relation to uniform loading of a transverse cable if m and n are interchanged.

All loading patterns which do not excite inextensional modes may be built up by superposition from uniformly loaded strings, and the associated displacements may be found by appropriate use of the expressions derived above in any given case. It is most instructive, however, to investigate two particularly simple special cases which may be assembled in this way.

Consider first a case in which one longitudinal string is loaded uniformly by joint loads W, as described above, while a second longitudinal string is loaded uniformly by forces $-W$, i.e.

by forces of the same magnitude but of opposite sense. A single parameter describes the tension in the transverse strings for this loading case, and it follows immediately by superposition of the preceding results that $\beta = 0$, i.e. that there is zero tension in the transverse strings. Thus each of the two longitudinal strings carries the load applied to it unaided, while the transverse and all other longitudinal strings are unstressed (apart, of course, from the initial stress in the system).

The key point about this particular case is that the two loaded longitudinal strings deform in such a way that the transverse strings undergo purely inextensional deformation. Thus no (extra) tension is developed in any of the transverse strings; and the two loaded longitudinal strings act essentially independently of the remainder of the net.

This idea may be extended readily to the general case in which each of the m longitudinal strings is loaded uniformly, but independently of the other, by nodal loads $W_i (i = 1 ... m)$. In any case where

$$\sum_1^m W_i = 0, \tag{16}$$

the transverse strings are not stressed on account of the applied loading, and each longitudinal string deflects as if it were an isolated string, i.e. according to (12) with $\beta = 0$, and with its own value of W. Note in particular that the displacement of each longitudinal cable is directly proportional to the load applied to it when (16) is satisfied.

Second, consider the case where all longitudinal strings are loaded uniformly by *equal* joint loads W, i.e.

$$W_i = W, \quad i = 1 ... m. \tag{17}$$

By superposition of the preceding results we find that each longitudinal string now carries tension on account of loads $(1 - m\beta) W$ acting on an isolated string, where β is given by (14). Thus we find, for a typical longitudinal string,

$$\begin{aligned} \sum w &= (n+1)(1-m\beta) Wl/AE\alpha^2 \\ &= \frac{n(n+1)(m+1)}{(2mn+m+n)} \frac{Wl}{AE\alpha^2}. \end{aligned} \tag{18}$$

The summation is over the n joints of the string. It follows that the mean joint displacement for the entire net is $(1/n)$ of this, so that

$$\bar{w} = \frac{(n+1)(m+1)}{(2mn+m+n)} \frac{Wl}{AE\alpha^2}. \tag{19}$$

Note that this expression is unaltered by the interchange of m and n, as indeed we expect for this case of uniform loading over the entire net.

Now if a particular pattern of loading were to produce nodal displacements such that the displacement of every joint was directly proportional to the load applied at the joint, we would describe it as an *eigenmode* of the system; and we would describe the common load/displacement factor as the corresponding *modal stiffness.*

Each of the special loadings cases (16) and (17) fulfills this condition, provided we overlook the fact that our calculation strictly gives the mean value, $\bar{w}$, of the nodal displacements of the string instead of the individual displacements.

With this proviso in mind, we may compute the mean modal stiffness K_{E1} for the case of uniformly loaded longitudinal strings satisfying (16) by putting $\beta = 0$ in (12):

$$K_{E1} = \frac{W}{\bar{w}} = \frac{n}{n+1} \frac{AE\alpha^2}{l}, \tag{20}$$

$$\simeq \frac{AE\alpha^2}{l} \tag{21}$$

for a large net.

In the case of uniform loading over the whole net, the corresponding quantity K_{E2} may be obtained from (19):

$$K_{E2} = \frac{W}{\bar{w}} = \frac{(2mn + m + n)}{(n+1)(m+1)} \frac{AE\alpha^2}{l} \tag{22}$$

$$\simeq \frac{2AE\alpha^2}{l} \tag{23}$$

for a large net.

In these expressions the subscript E denotes an extensional mode, and subscripts 1 and 2 distinguish the two different types of extensional mode. An expression similar to (20) may be written down for the case of uniformly loaded transverse strings with

$$\sum_1^n W_j = 0. \tag{24}$$

In either case the nodal stiffness is about one half of that for completely uniform loading. This reflects the fact that only one of the two sets of cables is operational when (16) or (24) is satisfied.

Altogether, then, there are $(m-1)$ independent cases of uniformly loaded longitudinal cables satisfying (16), (since the load on the nth cable is determined by the summation), $(n-1)$ independent cases for uniformly loaded transverse cables satisfying (24), and the single case (17) of uniform load at each nodal point of the net: $m+n-1$ cases in all. Thus, by means of our study of only two special cases we have determined the eigenmodes and the corresponding mean nodal stiffnesses for all of the "extensional" loading cases.

Throughout this section we have explicitly ignored any contribution which initial prestress may make to the net by virtue of small changes in geometry. We shall comment on this simplification in Section 11.

7. STIFFNESS OF INEXTENSIONAL MODES OF A PLANE NET

We examine next the mechanical behaviour of the net when it deforms inextensionally, in the way which we have described so far only in terms of kinematics. We shall develop the necessary physical ideas in stages, by means of a sequence of examples.

Let us investigate first the response of the two-bar system shown in Fig. 6(a) when a transverse force W is applied to the central joint. The bars are of length l and cross-sectional area A, the material has Young's modulus E, and there is an initial tension T_0 in the system. The abutments are rigid.

Consider the state of the assembly when the central joint is given a displacement w, small in the sense that $w/l \ll 1$. By elementary geometry we find that the strain in the bars $= 1/2(w/l)^2$ + terms of order $(w/l)^4$ and above, and hence that the tension T is now given by

$$T = T_0 + \tfrac{1}{2}AE(w/l)^2 + \ldots. \tag{25}$$

Resolution of the forces acting at the joint gives

$$W \simeq 2Tw/l \tag{26}$$

and hence, by (25),

$$W = 2T_0(w/l) + AE(w/l)^3 + \ldots. \tag{27}$$

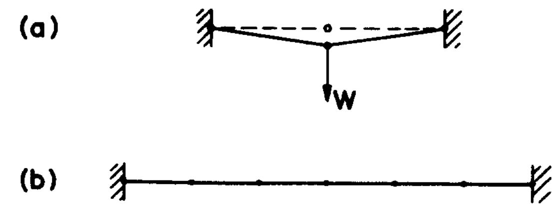

Fig. 6. Simple systems for the investigation of the stiffness of the inextensional modes of a single cable.

For sufficiently small displacements, therefore, the response of the system to the applied load is linear; and in this range the stiffness depends only on the initial prestress T_0 and not the elastic constant AE of the bar: see [5], Figs 4 and 5.

In this respect the system behaves quite differently from a conventional redundant structure under an arbitrary loading, where the stiffness of the system depends on the extensional stiffness of the bars but not on any initial stress which may be present on account of the redundancies.

For large displacements, of course, the cubic terms in w in expression (27) can become important. For example, the first and second terms on the r.h.s. of (27) are equal when the additional strain $\frac{1}{2}(w/l)^2$ is equal to the strain T_0/AE imparted to the originally stress-free bar when the prestress T_0 was applied. We shall return to considerations of this sort later on; but for the present we shall assume that the displacements are so small that we need consider only the linear part of the structural response of "inextensional" modes.

We are concerned here with "geometry change" effects: transverse loads are carried not primarily by changes in *tension* of the members but by small changes in *geometry* of the system. This is the key to the understanding of the stiffness of inextensional modes of cable nets.

Consider next the behaviour of the pretensioned string of bars shown in Fig. 6(b). Again there is an initial pretension of T_0, but now each of the n joints may be displaced independently a small distance w normal to the line of the original configuration. Let w_i be the displacement of the ith joint. By resolving the forces acting on a typical joint we find that the load acting on the ith joint, W_i, is given by

$$W_i = (T_0/l)(-w_{i-1} + 2w_i - w_{i+1}). \tag{28}$$

Hence in general we may write

$$\mathbf{W} = (T_0/l)\mathbf{M}\mathbf{w}, \tag{29}$$

where $\mathbf{W}$ and $\mathbf{w}$ are the column vectors W_i and w_i respectively, and $\mathbf{M}$ is the square matrix

$$\mathbf{M} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}. \tag{30}$$

It is convenient to discuss the behaviour of the system in terms of its eigenvectors and eigenvalues. It is easy to show that the n eigenvectors w_{iq} $(q = 1 \ldots n)$ of $\mathbf{M}$ are

$$w_{iq} = \sin\frac{q\pi i}{n+1}, \tag{31}$$

while the corresponding eigenvalues λ_q are

$$4\sin^2(q\pi/2(n+1)). \tag{32}$$

Thus the modal stiffness of the eigenmodes (31) of the system shown in Fig. 6(b) are

$$K_{Iq} = (T_0/l)\lambda_q = \frac{4T_0}{l}\sin^2\left(\frac{q\pi}{2(n+1)}\right),\ q = 1....n. \tag{33}$$

Here the subscript I denotes an inextensional mode. Figure 7 depicts the eigenmodes of a string of bars having 5 joints, together with the corresponding eigenvalue λ_q of $\mathbf{M}$.

In general, when n is large, the lower eigenvalues of $\mathbf{M}$ are approximately equal to $q^2T_0\pi^2/l(n+1)^2$: they are inversely proportional to the square of the half-wavelength of the modeform.

The eigenmodes and the corresponding nodal stiffnesses for the entire plane comparison net may be deduced easily from the above results. In order that the plane net shall model the curved net as well as may be, let every string have the same initial pretension T_0. Select any one of the n eigenmodes for the isolated longitudinal strings, and any one of the m eigenmodes for the isolated transverse strings, and arrange the amplitudes of these so that the two sets of strings intersect at every junction in the distorted configuration. At each node a transverse force is required for equilibrium of each of the two strings passing through the node; and since each of these separate forces is proportional to the displacement of the node, so also is the total force. Consequently this mode is an eigenmode of the whole system and the corresponding modal stiffness is the sum of the separate stiffnesses of the two string-modes.

There are n and m distinct eigenmodes for the longitudinal and transverse strings respectively. These thus generate the entire set of mn distinct eigenmodes of the complete plane net.

8. STIFFNESS OF SOME INEXTENSIONAL MODES OF A CURVED NET

We are now in a position to investigate the stiffness of the inextensional modes of the curved cable net. We have previously seen that there are $(m-1)(n-1)$ inextensional modes of this system, and that these may be generated from the $(n-1)$ and $(m-1)$ inextensional modes of the initially curved longitudinal and transverse strings, respectively.

In view of the preceding analysis of the plane comparison net, therefore, it seems clear that we should begin with an investigation of the eigenmodes of a single curved string. An examination of the condition of equilibrium of a shallow pretensioned curved string whose nodes have been displaced by small amounts from the initial configuration indicates that to a first approximation eqn (28) holds for the response of a node to an applied normal load. It follows that (29) also holds *provided* the displacement w_i now also satisfy the condition $\Sigma w_i = 0$. Inspection of (31) shows that the even ("skew-symmetric") eigenmodes ($q = 2, 4$, etc.) satisfy this condition, but that the odd ("symmetric") ones do not. Thus, approximately one quarter of the eigenmodes which we have already found for the plane comparison net are also

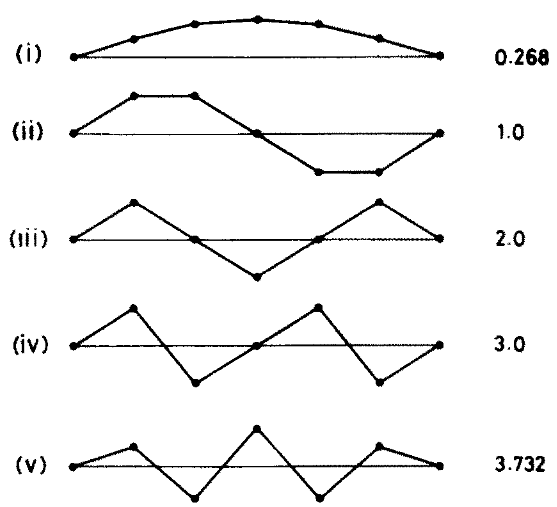

Fig. 7. Eigenmodes for the system of Fig. 6(b). The numbers are proportional to modal stiffness.

eigenmodes of the doubly-curved net. We shall briefly consider the remaining modes subsequently. But let us first investigate particularly the inextensional mode which has the lowest modal stiffness. It is clear that this is assembled from longitudinal and transverse string modes having $q = 2$, with an overall displacement pattern of the net in the form of a 2×2 chessboard. By use of (33) we may write down the modal stiffness of this mode:

$$K_{If} = \frac{4T_0}{l}\left\{\sin^2\left(\frac{\pi}{n+1}\right) + \sin^2\left(\frac{\pi}{m+1}\right)\right\}. \tag{34}$$

Or, on the assumption that $n \gg 1$ and $m \gg 1$,

$$K_{If} \simeq \frac{4\pi^2 T_0}{l}\left(\frac{1}{(n+1)^2} + \frac{1}{(m+1)^2}\right). \tag{35}$$

Here the subscript f denotes the fundamental inextensional mode of the curved net.

9. RELATIVE STIFFNESS OF EXTENSIONAL AND INEXTENSIONAL MODES

It will seem clear to anyone who has handled a simple model of a saddle-shaped cable net made from string or wire, that the least-stiff or softest inextensional mode is much less stiff than any extensional mode. It is of interest, therefore, to make a comparison between the nodal stiffness of the extensional and inextensional modes which we have studied, in order to discover what factors are involved in the determination of the relative stiffnesses.

For the sake of simplicity we shall consider a large net, with $n \gg 1$ and $m \gg 1$; and furthermore, in relation to the inextensional modes we shall consider a "square" net, with $m = n$. From (21) and (23) the nodal stiffness of the extensional modes are

$$K_E \simeq (2)AE\alpha^2/l. \tag{36}$$

Here, and in subsequent expressions, the 2 in parentheses is present for the uniform loading mode (denoted in (22) by subscript 2), but absent from the other modes (subscript 1). From (35) the modal stiffness of the most compliant inextensional mode is

$$K_{If} \simeq 8\pi^2 T_0/ln^2. \tag{37}$$

The ratio K_I/K_E is thus given by

$$\frac{K_I}{K_E} = \frac{8\pi^2}{(2)}\frac{T_0}{AE}\frac{1}{n^2\alpha^2}. \tag{38}$$

It is convenient now to define as ϵ_0 the elastic strain in a typical bar due to the initial prestress T_0. From (1):

$$\epsilon_0 = T_0/AE. \tag{39}$$

Equation (38) may then be written

$$\frac{K_I}{K_E} = \frac{2}{(2)}\frac{\epsilon_0}{\left(\dfrac{n\alpha}{2\pi}\right)^2}. \tag{40}$$

Now the total angle subtended by a longitudinal or transverse curved string with n nodes is $n\alpha$, so the expression $(n\alpha/2\pi)$ represents the fraction of a full circle which is occupied by one of the curved strings.

For example, if $n\alpha = 60°$, $n\alpha/2\pi = 1/6$ and

$$\frac{K_I}{K_E} = \frac{72}{(2)}\epsilon_0.$$

For a steel cable we might have $\epsilon_0 = 10^{-3}$, which gives $K_I/K_3 \simeq 1/30$, very approximately.

If this example is typical, the fundamental or most compliant inextensional mode of a cable net has a stiffness at least an order of magnitude less than that of the extensional modes.

It is important to realise, however, that the modal stiffnesses of the higher inextensional modes are considerably greater than those of the fundamental inextensional modes (see (33)). In the present example, therefore, many of the inextensional modes will be stiffer than the extensional modes, if the number of strings is large.

Let us investigate the parameters which determine the relative stiffnesses of the two kinds of mode. It is convenient to begin by devising a net for which the stiffness of the more compliant extensional modes is approximately equal to that of the fundamental inextensional mode. According to (4) this would be achieved if

$$\epsilon_0 \simeq \left(\frac{n\alpha}{2\pi}\right)^2 \simeq (n\alpha)^2/40. \tag{41}$$

Now suppose that each of the strings of the curved net of Fig. 1 is straight and just taut between the abutments when there is zero initial tension, and that the prestress is then imparted to it by pulling it into its basic curved configuration and connecting the nodes. Then, to a first approximation when $n\alpha \ll 1$ we have

$$\epsilon_0 \simeq (n\alpha)^2/24. \tag{42}$$

This is sufficiently close to the required condition (41) to give a simple physical interpretation of the criterion.

Typically in practice the "dip" or *sagitta* of a cable is about one-tenth of the span, so $n\alpha \simeq 1$. Hence, a cable net which satisfies (41) or (42) could readily be made from strings of material such as rubber, which is capable of extending by at least about 3% in the elastic range. But, clearly, the initial stretch of the steel cables of most practical tension-net structures is considerably less than this. Hence we may conclude that the most compliant inextensional mode in these structures will normally be considerably less stiff than the extensional modes.

Finally, we note that the dimensionless group which is represented by the r.h.s. of (38) and (40) is closely related to the group λ^2 which is used by Irvine[7] to characterise the behaviour of a hanging cable.

10. SYMMETRIC INEXTENSIONAL MODES

In Section 8 we remarked that the symmetric string-modes of the plane comparison net did not satisfy the inextensionality condition (9) of curved strings, and consequently that many of the inextensional modes of the plane net were not directly applicable to the curved net. In order to find the remaining inextensional modes of the curved net, it seems clear that we should begin by seeking the symmetrical eigenmodes of the inextensional deformation of a curved string. But here we encounter a paradox. There are in fact no "pure" eigenmodes of this kind. The nearest we can get to a symmetric eigenmode is a mode in which the nodal loads are proportional to the nodal displacements *plus* a small constant load. Thus, for modes of this sort we have

$$\mathbf{W} = (T_0/l)\mathbf{M}\mathbf{w} + \mathbf{c}, \quad \sum w = 0 \tag{43}$$

instead of (29), where $\mathbf{c}$ is a vector of constant elements. There is here a small interaction between inextensional and extensional modes, in the sense that the uniform loads c (which are necessary in order to balance the equation in the presence of the constraint $\Sigma w = 0$) are carried by an extensional mode.

It is not difficult to find the odd "eigenmodes" which satisfy (43) when

$$\mathbf{W} = K\mathbf{w}, \tag{44}$$

where K is a scalar modal stiffness. In general, for $q = 3,5\ldots$, they involve only a small departure from the corresponding modes in the straight string, and the value of K is always a little smaller than the unconstrained value. Figure 8 repeats the "straight string" modes of Fig. 7 for $q = 3$ and 5, and also shows the modes which satisfy (43) and (44) for a uniformly curved string. The case $q = 1$ does not fit into this pattern of course: there is no simple modification of (31) with $q = 1$ which will satisfy (43) and (44), and this mode is lost when we change from a straight string to a curved one: but see section 11.

The mode stiffnesses of the odd "eigenmodes" of curved strings are all a little less than those for straight strings. The largest discrepancy is about 8%, in the case n, $q = 5,3$. This is shown in Fig. 8, where the numbers correspond to modal stiffness, as in Fig. 7. The modification of the symmetric modes in order to enable them to meet the condition of inextensionality (9) of a curved string is analogous to a problem in the dynamics of suspended cable: see [7], chap.3.

11. COUPLING BETWEEN THE TWO FAMILIES OF MODES

We have discussed separately the mechanics of a cable net in its inextensional and extensional modes of deformation. We ought now to consider what happens when two patterns of loading in the two different classes are applied to the net simultaneously. Thus, for example, the net may sustain equal loads at all nodes, onto which is superposed a pattern of loading which excites the fundamental inextensional mode. Does the presence of the uniform load alter the nodal stiffness of the inextensional mode?

Since the stiffness of the inextensional modes depends on the tension in the strings working on geometry changes, and since the tension in the strings is altered by uniform loading of the net, the answer to this question in general is yes. However, the uniform load increases the tension in one set of strings and decreases the tension in the other set, so the contributions to the nodal stiffness of inextensional modes from the two sets of strings are to some extent self-cancelling. And indeed, we find that for a net having $n = m$, and the same value of α for the two sets of strings, the sum of the two sets of string tensions is independent of the magnitude of the uniform loading: hence in this case the nodal stiffness of the fundamental inextensional mode is quite unaffected by the uniform load.

The situation changes radically, of course, if the uniform loading is so large that one set of strings loses its tension altogether. These strings then go slack, the condition of inextensionality no longer applies to them and the other strings are free to execute an entirely different set of modes.

Another aspect of mode-interaction is involved when the loading of the net is purely uniform. The nodal stiffness of the uniform extensional mode has been calculated on the assumption that the changes of geometry of the net were insignificant in relation to the computation of tensions *via* the joint equilibrium equations. The geometry-change effect in this case is closely related to the fundamental inextensional mode of the plane comparison net. This mode was one of those which was not relevant to the inextensional deformation of a curved net; but it now reappears in association with the fundamental extensional mode. However, since the nodal stiffness of this mode is only about one quarter of that of the fundamental inextensional mode of the actual cable net, which itself has a small stiffness in comparison with the uniform extensional mode, it is clear that the "geometry-change" contribution to the stiffness of the fundamental extensional mode will usually be insignificant.

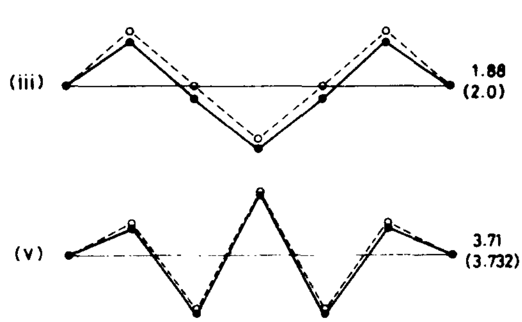

Fig. 8. Modified symmetric "eigenmodes" for a curved string, corresponding to cases (iii) and (v) of Fig. 7. The broken curves are taken from Fig. 7. For the meaning of the numbers, see text.

12. AN EXPERIMENT

Figure 9 shows the layout of a simple cable net $n, m = 2, 2$ which has been studied experimentally by Carstairs and Shipley as an undergraduate project. The cables were steel "piano wire" of dia. 0.38 mm, and they were soldered together at the nodes. The ends of the wires passed through small holes in a very stiff surrounding steel frame and were connected to standard guitar "machine-heads" mounted securely onto the frame. The horizontal distance between abutments was 533 mm, the vertical distance between the lower and upper abutment points was 116 mm, ant the horizontal square *ABCD* had sides of 153 mm.

The aim of the experiment was to observe the nodal stiffness of the net for two different loading patterns, and with various degrees of pretension of the cables. The two loading patterns were:

(i) "Skew": loads of equal magnitude at all four nodes, applied downwards at *A* and *C* and upwards at *B* and *D*.

(*ii*) "Uniform": equal vertical (downward) load at the four nodes.

Loads were applied by dead-weight, together with the use of pulleys for the upward-acting forces. Vertical deflections of the nodes were measured by four dial gauges connected to the nodes by suitable threads. Each dial gauge exerted a small and roughly constant force when its plunger was moving in one direction, and a force of a different value when moving in the other direction. The four dial gauges were of the same type, and they behaved equally in this respect. The dial gauges produced a relatively large hysteresis effect on the system when the direction of loading was changed; but since the well-defined *slopes* of the linear load/deflection curves were practically the same both on loading and on unloading in all tests, this hysteresis was not detrimental to the experiment.

The tension in the wires could be adjusted by turning the machine-heads. After the net had been set up accurately in its nominal configuration, all eight machine-heads were always turned in register. At any setting of the machine-heads the initial tension T_0 in the horizontal members was deduced from a separate experiment in which a vertical load was applied at the mid-point of one or more of the horizontal stretches of wire, and the corresponding node-stiffness was measured (see Fig. 6a). In all tests, including those for the determination of T_0, loads were increased in about six equal increments and then decreased. In all tests the load/deflection curves had a well-defined linear form.

The net was not "shallow", since the angle α subtended by wires passing through a node was about 16°. Nevertheless, the behaviour of the net was in broad agreement with the theory presented in this paper. Node-stiffnesses are plotted in Fig. 10 for the two distinct modes. The following points may be observed.

(a) The stiffness of the "skew" mode is directly proportional to T_0, in accordance with the theoretical prediction for an inextensional mode.

(b) The stiffness of the "uniform" mode is considerably greater than that of the inextensional mode, and relatively insensitive to the value of T_0. Note, however that it appears to vary linearly with T_0, in accordance with the analysis which has been sketched in Section 11.

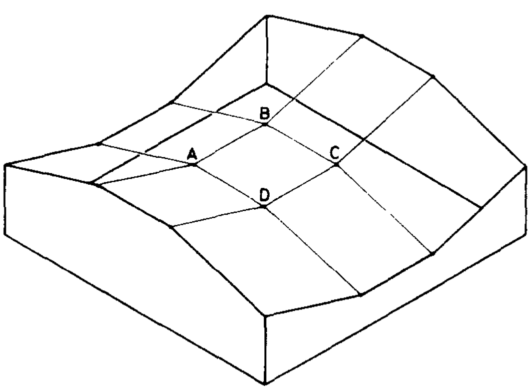

Fig. 9. Schematic view of the small-scale experimental net.

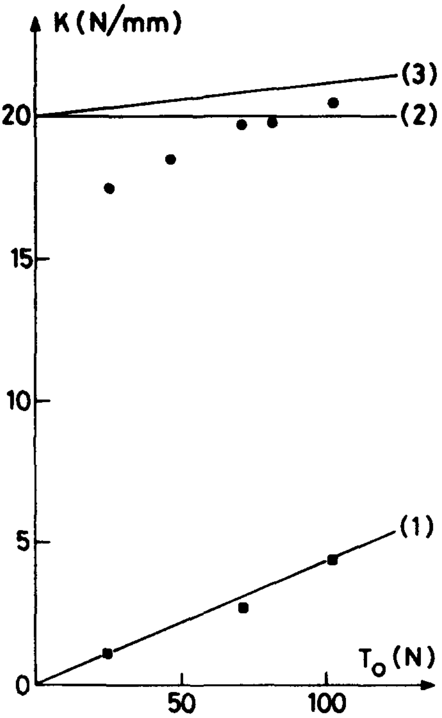

Fig. 10. Experimental observations. T_0 is the mean measured initial tension in the horizontal members, and K is the nodal stiffness, *viz.* the mean value of dW/dw for the four joints. ● symmetric loading, W at each node. ■ skew loading, W, $-W$, W, $-W$ at A, B, C, D. Line1: theoretical stiffness of the inextensional mode. Line 2: theoretical stiffness of extensional mode. Line 3: as Line 2 but taking into account also the geometry-change effect.

Also shown in Fig. 10 are the predictions of a simple theory like that of the present paper, but taking more accurate account of the trigonometry of the net. On the whole, the agreement is satisfactory. The reduced stiffness of the "uniform" mode observed for low values of initial tension may be due to extra compliance of the wire near the soldered joints on account of highly localised flexural effects.

13. CONCLUDING REMARKS

The simple net of Fig. 1 has been used as an example for a demonstration of the main features of behaviour of pretensioned cable nets. The contrast between the extensional and inextensional modes is particularly striking in this type of net.

It seems clear that all pretensioned cable nets involving two intersecting sets of cables will show broadly similar effects. But in systems where the main net is stretched between "edge" cables which are themselves anchored at only a few points (e.g. [8], p. 60), the number of extensional modes will clearly be a smaller fraction of the total number of modes than in nets of the kind shown in Fig. 1. And indeed, since it is unlikely that the special loading patterns which are necessary to excite the (pure) extensional modes will occur in practice, these particular modes will be of little practical significance.

The most novel feature of this paper is the *linear* analysis of the inextensional modes of the net. In relation to these modes, the net is not a *structure* in the usual sense, but a *mechanism*; and it is this feature which enables us to sidestep most of the complications inherent in the well-known field of analysis of "geometrically non-linear structures". The linear analysis of the inextensional modes is not valid for large deflections, for which the modes are not strictly inextensional, and involve changes in tension. The practical limits on the applicability of the linear analysis appear to depend on the level of *prestrain* in the wires, as shown in the simple example at the beginning of Section 7. It is possible that a point load applied to a single node of the net of Fig. 1 would exhaust the linear range of the inextensional modes, for a given level of prestrain, at smaller deflections than a more widely distributed pattern of loading. This is an area for further research. But the linear analysis should prove useful to engineers as a tool for investigating the behaviour of nets, in a first approximation, even for concentrated loads. An obvious area for application of the analysis is the study of the low-frequency modes of vibration of a net, which are at risk of excitation by fluctuating wind-loading.

In section 3 it was demonstrated that the curved net had only one degree of self-stress, but that there were more equations of equilibrium than were necessary for the determination of the bar-tensions in terms of the given tension in a single bar. It is easy to show that the excess of these equations is $(m-1)(n-1)$. These could be used to solve for the unique *configuration* of the unloaded net in terms of the $(m-1)(n-1)$ inextensional modes, if this were not known *ab initio*. This problem is generally known as "form-finding". We have not been concerned with it here, as the initial configuration of the net of Fig. 1 is obvious.

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